Saying It With Symbols

Quantifiers & First-Order Logic

Estimated Time: 60 minutes | 1-on-1 Session

Part 1: Where Do Numbers Live?

(approximately 10 minutes)

Before we can make precise mathematical statements, we need a way to say *which numbers we're talking about.* Mathematicians organize numbers into groups called **sets**.

Key Idea:

The Number Sets Here are the sets you'll use most often:

Symbol	Name	What's in it
$\mathbb{N}$	Natural numbers	$\{0, 1, 2, 3, 4, \ldots\}$
$\mathbb{Z}$	Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	Rational numbers	Fractions including whole numbers: $\frac{1}{2},\ -\frac{3}{4},\ 7,\ 0.25, \ldots$
$\mathbb{R}$	Real numbers	Everything on the number line (includes π, $\sqrt{2}$, etc.)

To say "x is a member of set S," we write $x \in S$. The symbol $\in$ means **"is in"** or **"belongs to."**
To say "x is *not* in set S," we write $x \notin S$.

Is It In the Set?

Q1. Write $\in$ or $\notin$ in each blank:

(a) 5 _____ $\mathbb{N}$
(b) -3 _____ $\mathbb{N}$
(c) -3 _____ $\mathbb{Z}$
(d) $\frac{1}{2}$ _____ $\mathbb{Z}$
(e) $\frac{1}{2}$ _____ $\mathbb{Q}$
(f) π _____ $\mathbb{Q}$

Q2. Every natural number is also an integer, and every integer is also a rational number. Fill in the blanks with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ to make each statement true:

(a) If $x \in \mathbb{N}$, then $x \in$ _______ (name a bigger set that also contains x)
(b) If $x \in \mathbb{Z}$, then $x \in$ _______
(c) The set that contains *all* numbers on the number line is _______

Q3. Write each statement using $\in$ or $\notin$:

(a) "7 is a natural number" ____________________
(b) "-10 is not a natural number" ____________________
(c) "$\sqrt{2}$ is a real number" ____________________
(d) "0.75 is a rational number" ____________________

Part 2: For All — The Universal Quantifier

(approximately 12 minutes)

Now that we can say *where* a number lives, let's make claims *about every number* in a set.

True for Everyone?

Q4. For each statement, decide if it's true for *every* natural number, or if you can find a counterexample:

(a) "If you add 1 to any natural number, you get another natural number." T / F
If false, counterexample: __________

(b) "Every natural number is positive." T / F
If false, counterexample: __________

(c) "If you square any integer, the result is positive." T / F
If false, counterexample: __________

Key Idea:

The Universal Quantifier (∀) The symbol ∀ means **"for all"** or **"for every."**
We use it to make a claim about *every* element in a set. Here's the anatomy of a universal statement:

$$\underbrace{\forall\, x}_{\text{"for every } x\text{"}}\ \underbrace{\in \mathbb{N}}_{\text{"in } \mathbb{N}\text{"}},\quad \underbrace{x + 1 \in \mathbb{N}}_{\text{the claim (predicate)}}$$

Read aloud: "For every x in the natural numbers, $x + 1$ is a natural number."

A **predicate** is a statement about x that is either true or false depending on what x is. In the example above, the predicate is "$x + 1 \in \mathbb{N}$."

Q5. Translate each symbolic statement into plain English:

(a) $\forall\, x \in \mathbb{Z}, \quad x + 0 = x$

(b) $\forall\, n \in \mathbb{N}, \quad n \cdot 1 = n$

Q6. Now go the other direction. Translate into symbols:

(a) "Every integer, when multiplied by zero, equals zero."

(b) "For every real number, adding it to its negative gives zero."

Hint: "its negative" means $-x$.

When "For All" Fails

Q7. To prove a "for all" statement is **false**, you only need *one* counterexample. Disprove each statement by finding one:

(a) $\forall\, x \in \mathbb{Z}, \quad x > 0$
Counterexample: $x =$ _______

(b) $\forall\, x \in \mathbb{N}, \quad x^2 > x$
Counterexample: $x =$ _______

(c) $\forall\, x \in \mathbb{R}, \quad \sqrt{x^2} = x$
Counterexample: $x =$ _______
Hint: Try a negative number.

Part 3: There Exists — The Existential Quantifier

(approximately 12 minutes)

"For all" makes a claim about *every* element. But sometimes we just want to say "*at least one* element has this property." That's what the existential quantifier does.

Can You Find One?

Q8. For each claim, find a specific number that makes it true:

(a) "There is a natural number that equals its own square."
Example: $x =$ _______ because ___________________

(b) "There is an integer that, when added to 5, gives 0."
Example: $x =$ _______

(c) "There is a rational number between 0 and 1."
Example: $x =$ _______

Key Idea:

The Existential Quantifier ($\exists$) The symbol $\exists$ means **"there exists"** or **"there is at least one."**

$$\underbrace{\exists\, x}_{\text{"there exists an } x\text{"}} \quad \underbrace{\in \mathbb{Z}}_{\text{"in } \mathbb{Z}\text{"}} \text{ such that } \quad \underbrace{x + 5 = 0}_{\text{the predicate}}$$

Read aloud: "There exists an integer x such that $x + 5 = 0$."

Key difference:

- $\forall$ claims something is true for *every* element.
- $\exists$ claims something is true for *at least one* element.

Q9. Translate each symbolic statement into English and decide if it's true or false:

(a) $\exists\, x \in \mathbb{N}$ such that $x + x = x$

English: ____________________________________

True or false? _________ If true, example: $x =$ _______

(b) $\exists\, x \in \mathbb{Z}$ such that $x^2 = -1$

English: ____________________________________

True or false? _________

Q10. Translate into symbols using $\exists$:

(a) "There is a real number whose square is 2."

(b) "Some integer, when multiplied by itself, equals 144."

Part 4: Exactly One — The Unique Existential

(approximately 8 minutes)

Sometimes "there exists" isn't precise enough. You don't just want to say *at least one* — you want to say *exactly one*.

How Many?

Q11. For each statement, find *all* solutions. How many are there?

(a) "There exists $x \in \mathbb{Z}$ such that $x + 3 = 7$."
Solution(s): $x =$ _________ How many? _______

(b) "There exists $x \in \mathbb{Z}$ such that $x^2 = 9$."
Solution(s): $x =$ _________ How many? _______

(c) "There exists $x \in \mathbb{N}$ such that $x^2 = 9$."
Solution(s): $x =$ _________ How many? _______

Notice how the set you're looking in can change how many solutions exist!

Key Idea:

The Unique Existential Quantifier ($\exists!$) The symbol $\exists!$ means **"there exists exactly one."** (Sometimes written $\exists\,!$ with a space.)

$$\exists!\, x \in \mathbb{Z} \text{ such that } x + 3 = 7$$

Read aloud: "There exists exactly one integer x such that $x + 3 = 7$."

This is *stronger* than $\exists$ — it says not only does a solution exist, but it's the *only one*.

Q12. For each statement, decide whether $\exists$ or $\exists!$ is more appropriate. Then write the full symbolic statement.

(a) "There is a natural number n such that $n + n = 8$."

Quantifier: ________ Statement: ________________________________

(b) "There is an integer whose square equals 25."

Quantifier: ________ Why? ________________________________

Part 5: Putting It All Together

(approximately 8 minutes)

Now let's translate between English and logic, choosing the right quantifier and set each time.

Two-Way Translation

Q13. Translate each statement into plain English:

(a) $\forall\, x \in \mathbb{R}, \quad x^2 \geq 0$

(b) $\exists\, n \in \mathbb{N}$ such that $n > 1{,}000{,}000$

(c) $\exists!\, x \in \mathbb{R}$ such that $x + x = x$

Q14. Translate each sentence into symbols:

(a) "Every natural number is greater than or equal to zero."

(b) "There exists a negative integer."

(c) "There is exactly one real number that, when you add it to any number, leaves that number unchanged." *(This is describing the number zero!)*

Challenge:

Two Variables Some statements involve *two* variables. When that happens, each variable gets its own quantifier.

Example: "For any two natural numbers, their sum is also a natural number."

$$\forall\, x \in \mathbb{N},\, \forall\, y \in \mathbb{N}, \quad x + y \in \mathbb{N}$$

Q15. Translate into symbols:

(a) "For every integer x and every integer y, the product $x \cdot y$ is an integer."

(b) "For every natural number, there exists a larger natural number."
Hint: The first variable uses $\forall$ *and the second uses* $\exists$*.*

Big Picture:

Why Does This Matter? Every line of mathematical proof uses these ideas. When a mathematician writes a theorem, they're using quantifiers to say *exactly* what they mean — no ambiguity, no confusion. The symbolic language you just learned is the same language used in university-level math, computer science, and philosophy. You can now read and write statements that are precise enough to build proofs on.

Homework: Quantifiers & First-Order Logic

Sets, Membership, and Mathematical Statements

Instructions: Show your reasoning. For true/false questions, provide an example or counterexample.

Section A: Sets and Membership

Q1. Write $\in$ or $\notin$ in each blank:

(a) 12 _____ $\mathbb{N}$
(b) -7 _____ $\mathbb{N}$
(c) -7 _____ $\mathbb{Z}$
(d) 0 _____ $\mathbb{N}$
(e) $\frac{3}{4}$ _____ $\mathbb{Z}$
(f) $\frac{3}{4}$ _____ $\mathbb{Q}$
(g) $\sqrt{2}$ _____ $\mathbb{Q}$
(h) $\sqrt{2}$ _____ $\mathbb{R}$

Q2. Write each statement using $\in$ or $\notin$:

(a) "-15 is an integer." ____________________

(b) "π is not a rational number." ____________________

(c) "0.333... (repeating) is a rational number." ____________________

Q3. True or false? If false, give a counterexample.

(a) Every element of $\mathbb{N}$ is also an element of $\mathbb{Z}$. T / F

(b) Every element of $\mathbb{Z}$ is also an element of $\mathbb{N}$. T / F

If false, counterexample: _______________

(c) Every element of $\mathbb{Q}$ is also an element of $\mathbb{R}$. T / F

Q4. For each number, write the *smallest* standard set ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$) it belongs to:

(a) 42: __________ (b) -5: __________ (c) $\frac{2}{3}$: __________ (d) $\sqrt{7}$: __________

Section B: The Universal Quantifier ($\forall$)

Q5. Translate each symbolic statement into plain English:

(a) $\forall\, x \in \mathbb{N}, \quad x + 0 = x$

(b) $\forall\, x \in \mathbb{R}, \quad x \cdot 1 = x$

(c) $\forall\, n \in \mathbb{Z}, \quad n + (-n) = 0$

Q6. Translate each sentence into a symbolic statement using $\forall$:

(a) "Every natural number is greater than or equal to 0."

(b) "For every real number, its square is greater than or equal to zero."

(c) "Any integer multiplied by 2 is an even number." *(Use "2n is even" as your predicate.)*

Q7. True or false? If false, provide a counterexample.

(a) $\forall\, x \in \mathbb{N}, \quad x^2 \geq x$ T / F

If false, counterexample: ______________

(b) $\forall\, x \in \mathbb{Z}, \quad 2x > x$ T / F

If false, counterexample: ______________

(c) $\forall\, x \in \mathbb{R}, \quad x + 1 > x$ T / F

If false, counterexample: ______________

Q8. Complete each statement with a predicate that makes it **true**:

(a) $\forall\, x \in \mathbb{N},$ ____________________________

(b) $\forall\, x \in \mathbb{Z},$ ____________________________

Q9. Write a "for all" statement that you think is true, then write one that is **false** and disprove it with a counterexample.

True statement: $\forall$ ______________________________________

False statement: $\forall$ ______________________________________

Counterexample: ___________________

Section C: Existential ($\exists$) and Unique Existential ($\exists!$)

Q10. Translate into English, then decide if each statement is true or false. If true, give an example.

(a) $\exists\, x \in \mathbb{N}$ such that $2x = 10$

English: ______________________________________

T / F Example: ______________

(b) $\exists\, x \in \mathbb{N}$ such that $x + 1 = 0$

English: ________________________________

T / F Why? ____________________

(c) $\exists\, x \in \mathbb{Z}$ such that $x + 1 = 0$

English: ________________________________

T / F Example: ____________

Q11. Translate into symbols:

(a) "There is a natural number whose square is 49."

(b) "Some real number, when multiplied by itself three times, gives −8."

Hint: "multiplied by itself three times" means x^3.

Q12. For each equation, find *all* solutions in the given set, then decide if $\exists$ or $\exists!$ is the right quantifier.

(a) $x + 7 = 10$ where $x \in \mathbb{Z}$

Solution(s): __________ Quantifier: $\exists$ / $\exists!$

(b) $x^2 = 16$ where $x \in \mathbb{Z}$

Solution(s): __________ Quantifier: $\exists$ / $\exists!$

(c) $x^2 = 16$ where $x \in \mathbb{N}$

Solution(s): __________ Quantifier: $\exists$ / $\exists!$

Q13. Translate into symbols using $\exists!$:

(a) "There is exactly one natural number n such that $n + n = 0$."

(b) "There is exactly one real number x such that $5x = 15$."

Q14. Consider the equation $x^2 = 1$.

(a) Write a **true** statement using $\exists!$ by choosing the right set:

(b) Write a **true** statement using $\exists$ (but where $\exists!$ would be false) by choosing a different set:

(c) What does this tell you about why the *set* matters in a mathematical statement?

Section D: Combining Quantifiers

Q15. Translate each statement into plain English:

(a) $\forall\, x \in \mathbb{N},\ \forall\, y \in \mathbb{N}, \quad x + y \in \mathbb{N}$

(b) $\forall\, x \in \mathbb{Z},\ \exists\, y \in \mathbb{Z}$ such that $x + y = 0$

(c) $\forall\, x \in \mathbb{N},\ \exists\, y \in \mathbb{N}$ such that $y > x$

Q16. Translate into symbols:

(a) "For every integer a and every integer b, the sum $a + b$ is an integer."

(b) "For any two natural numbers, their product is a natural number."

Q17. Translate into symbols (pay attention to which variable is "for all" and which is "there exists"):

(a) "Every integer has a negative." *(For every x, there exists a y such that $x + y = 0$.)*

(b) "For every natural number, there is a natural number that is double it."

Q18. True or false? If false, explain why.

(a) $\forall\, x \in \mathbb{N},\ \exists\, y \in \mathbb{N}$ such that $x + y = 10$

T / F Reasoning: ______________________________

(b) $\forall\, x \in \mathbb{N},\ \exists\, y \in \mathbb{Z}$ such that $x + y = 10$

T / F Reasoning: ______________________________

Section E: Challenge Problems

Q19. **(Challenge)** Translate each English statement into a precise symbolic statement. Choose the appropriate quantifier(s), variable(s), set(s), and predicate.

(a) "The sum of any two even numbers is even."

Hint: An even number can be written as $2m$ for some integer m. So you might say: for all $m \in \mathbb{Z}$ and for all $n \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $2m + 2n = 2k$.

(b) "Between any two different real numbers, there is a rational number."

Hint: You'll need two universal variables ($x, y \in \mathbb{R}$ with $x < y$) and one existential variable ($q \in \mathbb{Q}$).

(c) "There is no largest natural number."

Hint: Rephrase as "for every natural number, there exists a bigger one."

Q20. (Challenge) Create your own mathematical statements.

(a) Write a **true** statement that uses $\forall$, two variables, and the set $\mathbb{Z}$. Then explain in English why it's true.

(b) Write a **false** statement that uses $\forall$ and $\exists$ with two different variables. Then find a counterexample that disproves it.

Hint: Think about something that works for some numbers but not all. For example, is it true that for every $x \in \mathbb{N}$, there exists $y \in \mathbb{N}$ such that $y < x$?

(c) Write a statement that is true when the set is $\mathbb{Z}$ but **false** when the set is $\mathbb{N}$. Use any quantifier.

Hint: Think about negative numbers — they live in $\mathbb{Z}$ but not $\mathbb{N}$.

You can now read and write the same symbolic language used in every university math course on Earth.

Answer Sheet

Homework: Quantifiers & First-Order Logic

Section A: Sets and Membership

Q1. (a) $12 \in \mathbb{N}$ (b) $-7 \notin \mathbb{N}$ (c) $-7 \in \mathbb{Z}$ (d) $0 \in \mathbb{N}$

(e) $\frac{3}{4} \notin \mathbb{Z}$ (f) $\frac{3}{4} \in \mathbb{Q}$ (g) $\sqrt{2} \notin \mathbb{Q}$ (h) $\sqrt{2} \in \mathbb{R}$

Q2. (a) $-15 \in \mathbb{Z}$ (b) $\pi \notin \mathbb{Q}$ (c) $0.333\ldots \in \mathbb{Q}$ (it equals $\frac{1}{3}$)

Q3. (a) True. (b) False — counterexample: $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$. (c) True.

Q4. (a) $42 \in \mathbb{N}$ (b) $-5 \in \mathbb{Z}$ (c) $\frac{2}{3} \in \mathbb{Q}$ (d) $\sqrt{7} \in \mathbb{R}$

Section B: The Universal Quantifier

Q5.

(a) "For every natural number x, adding zero to x gives x." (The additive identity property.)

(b) "For every real number x, multiplying x by 1 gives x." (The multiplicative identity property.)

(c) "For every integer n, adding n to its negative gives zero." (The additive inverse property.)

Q6.

(a) $\forall\, x \in \mathbb{N}, \quad x \geq 0$

(b) $\forall\, x \in \mathbb{R}, \quad x^2 \geq 0$

(c) $\forall\, n \in \mathbb{Z}, \quad 2n$ is even

Q7.

(a) False. Counterexample: $x = 0$, since $0^2 = 0$ and $0 \not\geq 0$ is wrong — actually $0 \geq 0$ is true. Let's check $x = 0$: $0^2 = 0 \geq 0$. Check $x = 1$: $1 \geq 1$. This is actually **true** for all $x \in \mathbb{N}$.

(b) False. Counterexample: $x = -1$, since $2(-1) = -2$ and $-2 > -1$ is false. Also $x = 0$: $2(0) = 0 > 0$ is false.

(c) True for all real numbers. Adding 1 always makes a number strictly larger.

Q8. Many correct answers. Examples:

(a) $\forall\, x \in \mathbb{N}, \quad x + 1 \in \mathbb{N}$

(b) $\forall\, x \in \mathbb{Z}, \quad x + 0 = x$

Q9. Answers vary. Example true statement: $\forall\, x \in \mathbb{R},\ x + 1 > x$.

Example false statement: $\forall\, x \in \mathbb{Z},\ x > 0$. Counterexample: $x = -1$.

Section C: Existential and Unique Existential

Q10.

(a) "There exists a natural number x such that $2x = 10$." True. Example: $x = 5$.

(b) "There exists a natural number x such that $x + 1 = 0$." False — this would require $x = -1$, which is not in $\mathbb{N}$.

(c) "There exists an integer x such that $x + 1 = 0$." True. Example: $x = -1$.

Q11.

(a) $\exists\, x \in \mathbb{N}$ such that $x^2 = 49$

(b) $\exists\, x \in \mathbb{R}$ such that $x^3 = -8$

Q12.

(a) $x + 7 = 10 \Rightarrow x = 3$. Only one solution. $\exists!$

(b) $x^2 = 16 \Rightarrow x = 4$ or $x = -4$. Two solutions in $\mathbb{Z}$. $\exists$ (not $\exists!$)

(c) $x^2 = 16$ in $\mathbb{N} \Rightarrow x = 4$ only (since $-4 \notin \mathbb{N}$). One solution. $\exists!$

Q13.

(a) $\exists!\, n \in \mathbb{N}$ such that $n + n = 0$

(b) $\exists!\, x \in \mathbb{R}$ such that $5x = 15$

Q14.

(a) $\exists!\, x \in \mathbb{N}$ such that $x^2 = 1$ (only $x = 1$ works in $\mathbb{N}$)

(b) $\exists\, x \in \mathbb{Z}$ such that $x^2 = 1$ (both $x = 1$ and $x = -1$ work in $\mathbb{Z}$, so $\exists!$ would be false)

(c) The set determines which values are "available." The same equation can have one solution in one set and multiple solutions in another. The set is a crucial part of any mathematical statement.

Section D: Combining Quantifiers

Q15.

(a) "For every natural number x and every natural number y, the sum $x + y$ is a natural number." (Natural numbers are closed under addition.)

(b) "For every integer x, there exists an integer y such that $x + y = 0$." (Every integer has an additive inverse.)

(c) "For every natural number x, there exists a natural number y that is greater than x." (There is no largest natural number.)

Q16.

(a) $\forall\, a \in \mathbb{Z},\ \forall\, b \in \mathbb{Z},\quad a + b \in \mathbb{Z}$

(b) $\forall\, x \in \mathbb{N},\ \forall\, y \in \mathbb{N}, \quad x \cdot y \in \mathbb{N}$

Q17.

(a) $\forall\, x \in \mathbb{Z},\ \exists\, y \in \mathbb{Z}$ such that $x + y = 0$

(b) $\forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N}$ such that $m = 2n$

Q18.

(a) False. Counterexample: $x = 11$. There is no $y \in \mathbb{N}$ (i.e., $y \geq 0$) such that $11 + y = 10$, since that would require $y = -1$.

(b) True. For any $x \in \mathbb{N}$, choose $y = 10 - x$. Since y can be negative, and $y \in \mathbb{Z}$ allows negatives, this always works.

Section E: Challenge Problems

Q19.

(a) $\forall\, m \in \mathbb{Z},\ \forall\, n \in \mathbb{Z},\ \exists\, k \in \mathbb{Z}$ such that $2m + 2n = 2k$

This works because $2m + 2n = 2(m + n)$, so $k = m + n$, which is an integer.

(b) $\forall\, x \in \mathbb{R},\ \forall\, y \in \mathbb{R}$, if $x < y$ then $\exists\, q \in \mathbb{Q}$ such that $x < q < y$

This is the **density of the rationals** — between any two real numbers there's a rational.

(c) $\forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N}$ such that $m > n$

For any natural number n, we can pick $m = n + 1$.

Q20. Answers vary. Examples:

(a) True statement: $\forall\, a \in \mathbb{Z},\ \forall\, b \in \mathbb{Z}, \quad a + b = b + a$

"For any two integers, addition is commutative — the order doesn't matter."

(b) False statement: $\forall\, x \in \mathbb{N},\ \exists\, y \in \mathbb{N}$ such that $y < x$

Counterexample: $x = 0$. There is no natural number y such that $y < 0$.

(c) Example: $\exists\, x \in \mathbb{Z}$ such that $x < 0$

True in $\mathbb{Z}$ (e.g., $x = -1$). False if the set is changed to $\mathbb{N}$, since no natural number is negative.